OPTIMAL INVESTMENT NOTES

THEODOROS TSAGARIS

Key words. Portfolio Optimization, Duality theory, Martingale representation

1. Introduction. The theory of optimal investment dates back to the seminal paper of Markowitz [32] whose idea based on the examination of effective diversification of an investor's monetary resources. He asserts that maximization of only expected returns it is not sufficient to guarantee that the portfolio will have the smallest variance. Hence, he concludes that a trade-off between the expected return and the variance of the portfolio provides effective diversification and gives rise to the notion of attainable efficient combinations of mean and variance of the portfolio. The efficient attainable frontier is achieved when we maximize expected returns for given variance or minimize variance for given expected returns.

In the seminal papers of Merton [34, 33], a solution to the problem of optimal investment is presented for complete markets in the continuous case using tools of optimal stochastic control. Merton uses Hamilton-Jacobi-Bellman equation to derive a partial differential equation for the special case where asset prices are Markovian. He produces solutions for both finite/infinite horizon using exponential and logarithmic utility functions. Samuelson[39] has previously developed an equivalent theory for the discrete time case.

In 1973, Bismut [7] examines the optimal investment problem using a conjugate function in an optimal stochastic control setting. Bismut contribution is founded in the theory developed in his previous work [6], which in turn is inspired by [36], forms what is known now as the martingale approach in convex duality theory. Harrison and Pliska [20] provide a generic market model in continuous market trading and show that the optimal control problem can be solved using the martingale representation characterization of the terminal wealth by investing in complete markets. Cox, Huang [9] present sufficient conditions for the existence of solution to the consumption-portfolio problem under a certain class of utility functions. Karatzas, Lehoczky and Shreve [22] split the optimal investment problem into two more elementary problems where a small investor (one who cannot affect the market through his transactions) attempts to maximize the utility of consumption and terminal wealth separately.

Further studies by Cvitanic, Karatzas [12] generalize the results to the constrained case where the portfolio takes values in a predetermined closed, convex subset. Cuoco [11] extends the latter problem to the case of stochastic initial endowment. Karatzas, Lehoczky, Shreve and Xu [23] study the case where the number of stocks is strictly smaller than the dimension of Brownian motions (incomplete markets) and suggest a solution by introducing additional fictitious stocks to complete the market. Xu, Shreve [43] consider the finite horizon problem of optimal consumption/portfolio under short selling prohibition.

Kramkov, Schachermayer [26] study the problem of optimal investment in the more general framework of semimartingales, and provide necessary and sufficient conditions for the solution to the maximization of the expected utility of terminal wealth. They introduce the asymptotic elasticity of the utility function and show that it is necessary and sufficient condition for the optimal solution if it is strictly less than 1.

Transaction costs are an important financial feature whose omission overestimates the wealth process, and subsequently affects the theoretical or derivative prices of the underlying. Davis, Norman [16] have showed that the optimal policies are local times at the boundary, and that the boundary is determined by a nonlinear free boundary problem. Shreve, Soner [41] are influenced by Davis,Norman work to provide an analysis based on viscosity solutions to Hamilton-Jacobi-Bellman equation and additionally show that for a power utility $c^p/p,\ 0 < p < 1$ there is a correspondence between the value function and transaction cost in the order of $2/3$ power. Zariphopoulou [44] shows that the value function is a viscosity solution, in the sense of Crandall, Lions [10], of a system of variational inequalities with gradient constraints. Cvitanic, Karatzas [13] provide an analysis of hedging a claim under transaction costs using duality theory. Cvitanic, Pham, Touzi [14] apply [13] to show that the cheapest strategy to hedging is a buy and hold strategy of a call option. Kabanov, Last [21] deploy the more general semimartingale model with transaction costs in a continuous setting. Under assumption of properness of the solvency cone, they provide characterization of the initial endowments that allow investors to hedge contingent claims.

Lakner [28] approaches the optimal investment problem from the partial information case. He deploys some classical theory (Kalman-Bucy) to find an explicit solution for the portfolio process when the drift process is the unobservable component. Pham, Quenez [35] use stochastic duality and filtering theory to characterize the optimal portfolio process subject to the assumption the stock prices follow a stochastic volatility model.

2. Market Model and Problem Setup .

2.1. The Market Model. In this section, we define the financial market $\mathcal{M}$ consisting of d risky financial assets and a risk free money market account of rate r. We will consider the assets to be stocks, as has been traditionally modelled, though the formulations will not be restrictive to this class of assets. We assume that the stocks evolve continuously and that the market is complete i.e. that they are driven by d-dimensional Brownian motion. We define the notion of available information at time t in terms of σ-algebra and the conditions under which we ensure viability of the financial market. As we will see later, these conditions correspond to the conditions for existence of an equivalent martingale measure. We will follow closely the notation of Karatzas and Shreve [25].

Let us consider a model for an equity market defined as

$$dS_i(t) = S_i(t)[b_i(t) + \sum_{i=1}^{d} \sigma_{ij} dW_j(t)] \quad \forall t \in [0,T], \quad i,j = 1,\ldots,d \tag{2.1}$$

where

- $S_0(t)$ is a price share of the money market at time t
- $S_1(t),...,S_d(t)$ are the prices of d-dimensional stocks at time t
- $W(t)$ are d dimensional Brownian motions
- $b(t)$ are d dimensional mean rate of return processes subject to $\int_0^T ||b(t)||dt < \infty$
- $r(t)$ is the interest rate process at time t

The information available to investors up to time t is denoted by

$$\mathcal{F}^W(t) = \sigma\{W(s); 0 \le s \le t\}, \quad \forall t \in [0,T]$$

and is generated by $W(\cdot)$. The augmentation of the σ-algebra $\mathcal{F}^W$ is generated by the union of the null subsets $\mathcal{N}$

$$\mathcal{F}(t) = \sigma\{F^W(t) \bigcup \mathcal{N}\}, \quad \forall t \in [0,T]$$

We work on the context of augmented filtrations and we note that any $\mathcal{F}(t)$ measurable process $S(t)$, is right continuous semimartingale that admits a unique decomposition of a local continuous martingale and a right continuous with left hand limits finite variation process.

Assumption. The dispersion matrix $\sigma(t) = (\sigma_{dd})$ is adapted with respect to $\mathcal{F}(t)$ and bounded. The matrix $\sigma(t)$ has full rank, thelrefore is nonsingular and hence invertible.

Definition 2.1. *A financial market is called viable if there is a progressively measurable process $\theta(\cdot)$, such as the following equality holds*

$$b(t) - r(t)\mathbf{1} = \theta(t)\sigma(t), \quad \forall t \in [0,T] \tag{2.2}$$

and $\theta(\cdot)$ satisfies

$$\int_0^t ||\theta(s)||^2 ds < \infty$$

$$E[\exp\{-\int_0^t \theta^*(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\}] = 1$$

and the number of markets in the portfolio do not exceed the number of Brownian motion processes. The above definition ensures that there are no arbitrage opportunities in the market.

The discount process is defined as

$$\gamma(t) = \frac{1}{S_0(t)} = \exp\{-\int_0^t r(s)ds\} \tag{2.3}$$

and the exponential martingale or likelihood ratio process by

$$Z(t) = \exp\{-\int_0^t \theta^*(s)dW(s) - \frac{1}{2}\int_0^t ||\theta(s)||^2 ds\} \tag{2.4}$$

The state price density process is given by

$$H(t) = \gamma(t)Z(t) \tag{2.5}$$

3. The Portfolio and Consumption process. A portfolio process π is an $\mathbb{R}^d$-valued process and $c(\cdot)$ is the consumption process that takes values in the positive halfline. $\pi(\cdot)$ and $c(\cdot)$ are $\mathcal{F}(t)$ measurable and need satisfy

$$\int_0^T c(t)dt + \int_0^T \|\pi(t)\|^2 dt < \infty$$

The portfolio process $\pi(\cdot)$ is the proportion of wealth invested in $S_i(t)$ and it has support on $[0,1]$. Given some initial endowment $x > 0$, the wealth process associated to the portfolio weight π is denoted by X^π. The value of the portfolio invested in risky assets is given by $\sum_1^d \pi_i$ and the remaining proportion $1 - \sum_1^d \pi_i$ is invested in the risk-free deposit account. Before we show the formulation for $X^{\pi,c}$, we need to state some auxiliary results.

Set

$$\widetilde{W}(t) = W(t) + \int_0^t \theta(s)ds, \ 0 \le t \le T \tag{3.1}$$

$\widetilde{W}(t)$ is a Brownian motion under the measure

$$\widetilde{P} = E[Z(t)1_A], \ A \in \mathcal{F}(t) \tag{3.2}$$

and we say that $\widetilde{P}(A)$ and $P(A)$ are equivalent martingale measures under the filtration $\mathcal{F}(t)$.

Moreover, if θ satisfies the Novikov condition

$$E[\exp\{\frac{1}{2}\int_0^t ||\theta(s)||^2 ds\}] < \infty$$

then it follows from Girsanov that $\widetilde{W}(t)$ is a martingale under measure $\widetilde{P}$.

Assume there is an agent who can decide on a non-anticipative basis with respect to $\mathcal{F}(t)$, whether he deposits his capital in stock $S_i, 1 \le i \le d$ or in the risk-free deposit S_0. The agent is also allowed to withdraw funds at a rate of consumption $c(t), t \ge 0$. Then, assuming that the investor's transactions in the market are sufficiently small, in the sense that do not "move" the market, we define the wealth process as follows:.

$$dX^\pi(t) = \sum_{i=1}^d \pi_i(t)X^\pi(t)\left(b_i(t)dt + \sum_{j=1}^d \sigma_{ij}dW_j(t)\right) + \left(1 - \sum_{i=1}^d \pi_i(t)\right)X^\pi(t)r(t)dt - c(t)dt \tag{3.3}$$

$$= \sum_{i=1}^{d} \pi_i(t) X^\pi(t) \left(b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij} \left(d\widetilde{W}_j(t) - \theta_j(t)dt \right) \right) + \left(1 - \sum_{i=1}^{d} \pi_i(t) \right) X^\pi(t) r(t) dt - c(t)dt$$

$$= \left(X^\pi(t) r(t) - c(t) \right) dt + \sum_{i=1}^{d} \pi_i(t) X^\pi(t) \sum_{j=1}^{d} \sigma_{ij} d\widetilde{W}_j(t) \tag{3.4}$$

or in the equivalent vector formulation

$$= \left(X^\pi(t) r(t) - c(t) \right) dt + X^\pi(t) \pi_i^*(t) \sigma(t) d\widetilde{W}(t)$$

PROPOSITION 3.1. *The process*

$$\gamma(t) Z(t) X^\pi(t) = H(t) X^\pi(t) = x + \int_0^t H(t) X^\pi(t) (\sigma^*(t)\pi(t) - \theta(t))^* d\widetilde{W}(t) \tag{3.5}$$

is a supermartingale.

Proof. We initially obtain an sde by Ito's lemma for $f(t, X^\pi) = \gamma(t) X^\pi(t)$ and we proceed by applying Ito's product rule on $dZ(t) f(t, X^{\pi,c})$ to obtain (3.5). We note that (3.5) is a local martingale and by Fatou's lemma any non-negative local martingale is a supermartingale. □

DEFINITION 3.2. *A portfolio strategy π is called admissible if, for any initial endowment $x > 0$, we have*

$$X^{\pi,c}(t) \geq 0, \quad \forall t \in (0, \infty)$$

The class of admissible strategies is denoted by $\mathcal{A}(x)$

4. Utility Functions. A utility function is a strictly increasing concave function that satisfies

$$U'(0+) = \lim_{x \downarrow 0} U'(x) = \infty, \quad U'(\infty) = \lim_{x \to \infty} U'(x) = 0$$

The inverse of U' is denoted by I and satisfies

$$I(0+) = \infty, \quad I(\infty) = 0$$

DEFINITION 4.1.
The convex dual of U is

$$\widetilde{U}(y) = \sup_{x>0} \{ U(x) - xy \}, \quad y \in \mathbb{R} \tag{4.1}$$

where $\widetilde{U}(y)$ is convex and strictly decreasing and satisfies

$$U(I(y)) \geq U(x) + y[I(y) - x], \forall x > 0, y > 0 \tag{4.2}$$

$$\widetilde{U}(U'(x)) \leq \widetilde{U}(y) - x[U'(x) - y], \forall x > 0, y > 0 \tag{4.3}$$

For further details the reader could consult [1, 5, 8, 6, 17, 37, 36]

5. Portfolio Optimization: Martingale Approach .

5.1. The Optimization Problem. In this section we provide existence results for the optimal portfolio process $\widehat{\pi}(t)$, and show an explicit formulation under the logarithmic utility case. We follow similar arguments as in [12, 22, 23, 25, 43].

For a given endowment $x > 0$ and a utility function U, we need to maximize the expected utility from terminal wealth $EU(X^{\pi})$ over the class $\mathcal{A}(x)$ of all admissible portfolio strategies π that satisfy

$$E\left[U(X^{\pi}(T))\right] < \infty \tag{5.1}$$

where $f^{-} = \max\{-f, 0\}$. The value function is given by

$$V(x) = \sup_{\pi \in \mathcal{A}(x)} EU(X^{\pi}(T)) \tag{5.2}$$

5.2. Solution to the terminal wealth problem. Define the following function

$$\mathcal{X}(y) = E\left[\int_0^T H(T)I(yH(T))\right] 0 < y < \infty$$

From monotone convergence theorem and the monotonicity of $I(\cdot)$, we deduce that $\mathcal{X}(y)$ is continuous, strictly decreasing mapping from $(0,\infty) \to (0,\infty)$.

Now, define a non-negative measurable variable with respect to $\mathcal{F}(T)$ that for every portfolio $\pi \in \mathcal{A}(x)$, the supermartingale $H(t)X^{\pi}(t)$ satisfies

$$E[H(T)X^{\pi} \leq x]$$

The unconstrained optimization problem reduces to

$$E[U(H(\xi))) + y(x - E(H(T)\xi))$$

$$= xy + E[U(H(\xi)) - H(T)\xi]$$

$$\leq xy + E(\widetilde{U}[yH(T))]$$

Therefore, the value that maximizes the terminal wealth is given by

$$\xi = I(\mathcal{Y}(x)H(T)) \tag{5.3}$$

,where $\mathcal{Y}$ is the strictly decreasing inverse mapping of $\mathcal{X}$ from $(0,\infty)$ onto itself.

Theorem 5.1. *The optimal portfolio process is given by*

$$\widehat{\pi}(t) = \frac{1}{H(t)\widehat{X}^{\widehat{\pi}}(t)}(\sigma^*(t)^{-1}(\psi(s) + M(s)\theta(s))^* d\widetilde{W}(t)) \tag{5.4}$$

with $\widehat{X}^{\widehat{\pi}}(0) = x$ *and* $\widehat{X}^{\widehat{\pi}}(T) = \xi$.

Proof. Define $M(t) = E(H(T)\xi|\mathcal{F}(t))$

$M(t)$ admits a stochastic representation satisfying

$$M(t) = x + \int_0^t \psi^*(s)dW(s)$$

given that the condition

$$||\psi^*(s)||^2 < \infty$$

is satisfied.

With the use of Ito's lemma, we deduce

$$d(M(t)\frac{1}{Z(T)}) = \frac{1}{Z(T)}(\psi^*(t) + M(t)\theta(t))^* d\widetilde{W}(s)$$

Hence,

$$\gamma(T)\xi = \frac{M(T)}{Z(T)} = x + \int_0^T \frac{1}{Z(s)}(\psi(s) + M(s)\theta(s))^* d\widetilde{W}(s)$$

$$\widehat{X}^{\widehat{\pi}}(T) = \frac{M(T)}{H(T)} = \frac{1}{\gamma(T)}\left[x + \int_0^T \frac{1}{Z(s)}(\psi(s) + M(s)\theta(s))^* d\widetilde{W}(s)\right]$$

(The result follow by comparing the latter equation to equation 3.5).

□

5.3. Examples. In the case where the utility function is logarithmic, we have from (5.3),

$$\xi = x\exp\left[\int_0^T \left(r(t) + \frac{1}{2}\|\theta\|^2\right) + \int_0^T \theta^*(t)dW(t)\right]$$

From (3.5), we deduce that

$$\widehat{X}^{\widehat{\pi}}(t) = \xi = \frac{x}{H(T)}$$

iff the portfolio process is given by

$$\widehat{\pi} = (\sigma(t)\sigma^*(t))^{-1}(b(t) - r(t)\mathbf{1})$$

.

Substituting the above in 5.2, we have

$$V(x) = E\int_0^T \left(r(t) + \frac{1}{2}\|\theta\|^2\right) + \log x$$

assuming that the local martingale integral part vanishes under expectation.

6. Merton Portfolio Problem . This section is devoted to the seminal paper of Merton [33] (see [18, 19, 25] as well) who examines both the portfolio and consumption model for an individual in the continuous time case. Merton derives the optimal portfolio process in the multiasset case where the rate of returns are generated by a Brownian motion model, and examines in particular the two asset case with constant relative risk aversion.

We proceed by referring to [22] for a solution with more general utility function than HARA (Hyperbolic Absolute Risk Aversion) used by Merton. Karatzas, Lehoczky, Shreve show that the value function $V(x)$ satisfied by the dynamic programming equation can be linearized by introducing two linear differential functions with Feynman-Kac representation. Then, the optimal consumption is obtained by arguments involving the Legendre conjugate, and the optimal investment process is deduced correspondingly from the dynamic programming equation.

We draw the attention of the reader to the fact, that the dynamic programming approach assumes Markovian nature. In particular, we use controlled diffusion processes where the dynamic programming equation turns to be a nonlinear second order partial differential equation. Moreover, we assume that value function satisfied by the dynamic programming equation has some necessary smoothness properties, although it is often the case as in deterministic optimal control that value functions lack smoothness. The latter problem is overcome if the value function is interpreted as viscosity solution. The interested reader could consult [10] for general theory of viscosity solutions of Hamilton-Jacobi equations.

In order to avoid repeating in the text the necessary conditions for the existence of solution to the dynamic problem, we will develop the 2-asset case in a less rigorous way, which will in turn motivate the more formal general case.

6.1. The original 2-asset optimal consumption problem. In this subsection, we slightly deviate from the multi asset notation we have established in order to introduce the original 2-asset model.

Assume that an investor has a portfolio of two assets, one risk-less asset that evolves according to

$$dB(t) = B(t)rdt$$

and a risky asset that evolves according to

$$dS(t) = S(t)\left(bdt + \sigma dW(t)\right)$$

Let b be the expected rate of return for the risky asset and r the risk-less rate, and assume that $b > r$ holds. Moreover, we denote the wealth of the investor at time t by $X^{\pi,c}(t)$, where the superscript denotes dependence on the portfolio allocation rule π and consumption rate c. Finally, we assume throughout that $X^{\pi,c}(t) \geq 0$ and $c(t) \geq 0$, where c denotes the consumption rate.

Let Z be the amount in monetary terms invested in the bank account and Y the amount invested in equities. Moreover, let ℓ be the amount transferred from the bank account to acquire equities and m the converse. Then, the SDE's for Z and Y are as follows:

$$dZ(t) = rZ(t)dt - c(t)dt + dM(t) - dL(t)$$

$$dY(t) = Y(t)(bdt + \sigma dW) + dL(t) - dM(t)$$

where $L(t) = \int_0^t \ell(s)ds$ and $M(t) = \int_0^t m(s)ds$.

Letting $X(t) = Z(t) + Y(t)$, then he wealth process of the investor turns to be

$$dX^{\pi,c}(t) = X^{\pi,c}(t)rdt + \pi(t)(b-r)dt + \pi(t)X^{\pi,c}(t)\sigma dW(t) - c(t)dt \tag{6.1}$$

where W is one-dimensional Brownian motion process and $\pi(t) = \frac{Y(t)}{X(t)}$. Note that the above formulation asserts that ratio of risk-less to risky assets is given by

$$\frac{Z(t)}{Y(t)} = \frac{Z(t)+Y(t)}{Y(t)} - 1 = \frac{X(t)-Y(t)}{Y(t)} = \frac{1-\pi(t)}{\pi(t)}$$

The ratio $\frac{\widehat{\pi}}{1-\widehat{\pi}(t)}$ that dictates the optimal trade-off between bank-account and equities holdings is known as "Merton line".

Now, suppose that the controls (π, c) are admissible and have compact support in $\mathbb{R} \times \overline{\mathbb{R}}_+$, where $\overline{\mathbb{R}}_+$ denotes the non-negative halfline. We also assume that the maps $(t,x) \to \pi(t,x)$, $(t,x) \to c(t,x)$ from $[0,T] \times \Omega$ into $\mathbb{R} \times \overline{\mathbb{R}}_+$ are $\mathcal{B}([0,T]) \times \mathcal{F}(t)$ measurable for $t \in [0,T]$. We want to maximize the total expected utility of consumption, discounted at β

$$V(x;\pi,c) = \sup_{\pi,c\in U} J(x;\pi,c) \tag{6.2}$$

$$J(x;\pi,c) = E_x\left[\int_0^\tau e^{-\beta s}U(c(s))ds\right] \tag{6.3}$$

where τ is the stopping time when $X^{\pi,c}$ fails the admissibility criterion of $X^{\pi,c}(t) > 0$. We note from the properties of utilities functions defined in 4, that $V(x;\pi,c)$ is monotonically increasing in $[0,\infty)$.

Then, the HJB can be formulated as follows (see [2, 4, 3, 18, 19, 27] for detailed assumption and conditions on HJB formulation):

$$\mathcal{J}(x;\pi,c) = \frac{\vartheta G}{\vartheta t} + \sup_{\pi\in U}\left[\frac{x^2\sigma^2\pi^2}{2}G_{xx} + (b-r)x\pi G_x\right] + xrG_x + \sup_{c\in U}[U(c) - cG_x] = 0 \tag{6.4}$$

Maximizing the quantities in the square brackets, we deduce

$$\widehat{\pi} = -\frac{(b-r)G_x}{x\sigma^2 G_{xx}}, \qquad U_c(c) - G_x = 0 \tag{6.5}$$

For $G(x) = kx^\delta$ and $U(c) = \frac{1}{\delta}c^\delta,\ 0<\delta<1$, we deduce

$$\widehat{\pi} = \frac{b-r}{(\delta-1)\sigma^2}, \qquad \widehat{c} = (\delta k)^{\frac{1}{\delta-1}}x$$

where k after some elementary but tedious calculations is given by

$$k = \left(\beta - \frac{(b-r)^2k^*}{\sigma^2}\right)^{(\delta-1)} k^{**}$$

where

$$k^* = \frac{\delta}{2(\delta-1)}, \qquad k^{**} = \left(\frac{1}{1+\delta^{\delta/(\delta-1)}}\right)^{(\delta-1)}$$

For the logarithmic utility case, we obtain the following partial differential equation

$$-\ln(G_x) - \frac{1}{2}\phi^2\frac{(G_x)^2}{G_{xx}} - 1 + G_x rx - \beta G = 0$$

For an explicit solution to the above PDE, we refer to [15] who obtains

$$G(x) = \frac{r + \frac{1}{2}\phi^2 - \beta + \beta\ln(\beta)}{\beta^2} + \frac{1}{\beta}\ln(x) \tag{6.6}$$

and the optimal strategies are

$$\pi = \frac{\phi}{\sigma}, \quad c(t) = \beta X(t)$$

where $\phi = \frac{b-r}{\sigma}$.

6.2. The general case. In this section we define and solve the stochastic optimal control problem of maximizing the expectation of discounted consumption of the investor under the general utility functionals defined in section (4). We assume that the quantities $b(t), r(t), \sigma(t)$ defined above, are now constants. Our exposition closely follows arguments found in [22, 25, 24].

Subject to a non-negative initial endowment x, and $(\pi,c)\in\mathcal{A}(x)$, we define our "running cost" function

$$J(t,x;\pi,c) = E\int_0^T e^{-\beta t}U(c(t))ds$$

where T is the terminal time, and β the discount constant. The objective is to maximize $J(x;\pi,c)$ over all admissible controls (π,c)

$$V(t,x;\pi,c)=\sup_{\pi,c\in\mathcal{A}(x)} J(x;\pi,c) \tag{6.7}$$

By applying Ito's lemma in $e^{-rt}X^{\pi,c}(t)$, where $X^{\pi,c}(t)$ is given in (3.4), we deduce

$$\gamma(T)X^{\pi,c}(T)-x=\int_0^T \gamma(t)X^{\pi,c}(t)\pi^*(t)\sigma(t)d\widetilde{W}(t)-\int_0^T \gamma(t)c(t)dt$$

It is standard to define the following

$$E\int_0^T Z(t)c(s)ds\le x \tag{6.8}$$

The above is called budget constraint, and the constraint is satisfied when the inequality is replaced by equality. Moreover, as it is indicated by the value function (6.7) and the above constraint, on the following text we will only focus on the admissible control c.

DEFINITION 6.1. *Denote by $\mathcal{D}(x)$ the class of consumption processes for which the following holds:*

$$E\int_0^T e^{-rt}c(t)dt=x$$

The Merton problem is equivalent to the constrained problem of maximizing the expected discounted utility under the budget constraint (6.8). The constrained problem turns unconstrained with the help of the Lagrange multiplier y

$$\mathcal{L}(t,x,y,c)=E\int_0^T e^{-\beta t}U(c(t))dt+y\left(x-E\int_0^T e^{-rt}Z(t)c(t)dt\right)$$

$$\mathcal{L}(t,x,y,c)=xy+E\int_0^T e^{-\beta t}\left[\left(U(c(t))-e^{(\beta-r)t}Z(t)c(t)\right)dt\right]$$

Maximizing the above quantity with respect to c, we obtain the following inequality

$$\mathcal{L}(t,x,y,c)\le xy+E\int_0^T e^{-\beta t}\left[\left(U(\widehat{c}(t,y))-e^{(\beta-r)t}z(t)\widehat{c}(t,y)\right)dt\right]$$

which turns equality if and only if

$$\widehat{c}(t,y(x))=I(y(x)e^{(\beta-r)t}Z(t))$$

and subject that the Lagrange multiplier y saturates the budget constraint inequality.

DEFINITION 6.2. *Let the strictly continuous function*

$$\mathcal{X}(t,y)=E\int_0^T \gamma(t)Z(t)\widehat{c}(\mathcal{Y}(t,x))dt \tag{6.9}$$

defined in $\overline{\mathbb{R}}^+\to\overline{\mathbb{R}}^+$ and let the strictly decreasing map $\mathcal{Y}:\overline{\mathbb{R}}^+\to[0,\overline{y}]$ be the inverse image of $\mathcal{X}$, where $\overline{y}=\sup\{y\ge 0\}$, and $\widehat{c}$ is given by

$$\widehat{c}(\mathcal{Y}(t,x))=I(\mathcal{Y}(t,x)e^{(\beta-r)t}Z(t))$$

Theorem 6.3. *(Karatzas, Lehoczky, Shreve [22]) The consumption process $\widehat{c}(t, \mathcal{Y}(t,x))$ is optimal, and the following function holds*

$$V(t, x; \widehat{c}) = \sup_{\widehat{c} \in \mathcal{A}(x)} J(x; \pi, \widehat{c}(\mathcal{Y}(t,x))) \tag{6.10}$$

Proof. The proof follows by comparing the quantities $\widehat{c}(\mathcal{Y}(t,x))$ and $c(t)$. We have,

$$E \int_0^T e^{-\beta t} \left[U(\widehat{c}(\mathcal{Y}(t,x))) - U(c) \right]$$

It follows from 4.2,

$$E \int_0^T e^{-\beta t} \left[\left(U(\widehat{c}(\mathcal{Y}(t,x))) - \mathcal{Y}(t,x) e^{(\beta - r)t} Z(t) \widehat{c}(\mathcal{Y}(t,x)) \right) - \left(U(c(t)) - \mathcal{Y}(t,x) e^{(\beta - r)t} Z(t) c(t) \right) \right] dt$$

$$= E \int_0^T e^{-\beta t} \left[U(\widehat{c}(\mathcal{Y}(t,x))) - U(c(t)) \right] dt + \mathcal{Y}(t,x) E \int_0^T Z(t)(c(t) - \widehat{c}(\mathcal{Y}(t,x))) dt$$

$$+ \mathcal{Y}(t,x) E \int_0^T e^{-rt} Z(t) \left[c(t) - \widehat{c}(\mathcal{Y}(t,x)) \right] dt$$

The second and third term vanish since they belong to the same class, and first term is non-negative through concavity and monotonicity of U, which proves the claim.
□

From the optimality of $\widehat{c}(\mathcal{Y}(t,x))$, we deduce the value function

$$\widehat{V}(t, x; \pi, \widehat{c}) = \sup_{\pi, \widehat{c} \in \mathcal{A}(x)} J(x; \pi, \widehat{c}(\mathcal{Y}(t,x))) \tag{6.11}$$

$$\widehat{J}(t, x; \pi, \widehat{c}) = E \int_0^T e^{-\beta t} U(\widehat{c}(\mathcal{Y}(t,x))) ds \tag{6.12}$$

We change the variable $y = \mathcal{Y}(t,x)$ and by applying Ito's lemma on

$$W(t,y) = y(x) e^{(\beta - r)t} Z(t)$$

we obtain

$$dW(t,y) = W(t, y[(b - r\mathbf{1})dt - \theta^* dW(t)] \tag{6.13}$$

and by using the Feynman-Kac formula (see in appendix A), we obtain the following linear differential equation

$$-J_t(t,y) + \beta J(t,y) - (\beta - r) y J_y(t,y) - \frac{1}{2} \|\theta\|^2 y^2 J_{yy}(t,y) - U(I(y)) = 0 \tag{6.14}$$

The above Cauchy problem can be solved explicitly under some continuity and growth conditions (see [22]).

We proceed by rewriting the dynamics of (6.13) under the risk neutral measure $\widetilde{P}$ by applying (3.1) in (6.13) to obtain

$$dW(t,y) = W(t, y[(b - r\mathbf{1} + \|\theta\|^2)dt - \theta^* d\widetilde{W}(t)] \tag{6.15}$$

Again using Feynman-Kac formula, we obtain a linear differential equation for (6.9)

$$-\mathcal{X}_t(t,y)+r\mathcal{X}(t,y)-(\beta-r+\|\theta\|^2)y\mathcal{X}_y(t,y)-\frac{1}{2}\|\theta\|^2y^2\mathcal{X}_{yy}(t,y)-I(y)=0 \tag{6.16}$$

We note that the following equations hold:

$$J_y(t,y)=y\mathcal{X}(t,y)$$

$$V_x(t,x)=e^{-\beta t}\mathcal{Y}(t,x)$$

$$\mathcal{Y}_x(t,x)=\frac{1}{\mathcal{X}_y(t,y)}$$

$$\mathcal{Y}_t(t,x)=-\mathcal{Y}_x(t,\mathcal{X}(t,y))\mathcal{X}_t(t,y)=-\frac{\mathcal{X}_t(t,y)}{\mathcal{X}_y(t,y)}$$

Applying HJB equation (see appendix A) on the value function $\widehat{V}$ and using the above formulations, we obtain

$$V_t(t,x)+\sup_{\widehat{\pi},\widehat{c}\in\mathcal{A}}\left(\frac{1}{2}\|\sigma^*\pi\|^2V_{xx}(t,x)+\left[(rx-c)+\pi^*(b-r\mathbf{1})\right]V_x(t,x)+e^{-\beta t}U(c(t))\right)=0$$

$$e^{-\beta t}\left[J_t(t,\mathcal{Y}(t,x))-\beta J(t,\mathcal{Y}(t,x))+J_y(t,\mathcal{Y}(t,x))\mathcal{Y}_t(t,\mathcal{X}(t,y))\right]$$

$$+e^{-\beta t}\left[\sup_{\widehat{\pi},\widehat{c}\in\mathcal{A}}\left(\frac{1}{2}\|\sigma^*\pi\|^2\mathcal{Y}_x(t,x)+\left[(rx-c)+\pi^*(b-r\mathbf{1})\right]\mathcal{Y}(t,x)+U(c(t))\right)\right]=0 \tag{6.17}$$

Maximizing the quantity enclosed in square brackets for π and c, we obtain

$$\widehat{c}=I(\mathcal{Y}(t,x))$$

$$\widehat{\pi}=-(\sigma\sigma^*)^{-1}(b-r\mathbf{1})\frac{\mathcal{Y}^2(t,x)}{\mathcal{Y}_x(t,x)}$$

Substituting the above quantities in the HJB equation (6.17) and by properties of utility functions (4), we have

$$e^{-\beta t}\left[J_t(t,\mathcal{Y}(t,x))-\beta J(t,\mathcal{Y}(t,x))+J_y(t,\mathcal{Y}(t,x))\mathcal{Y}_t(t,\mathcal{X}(t,y))\right]$$

$$+e^{-\beta t}\left[r\mathcal{X}(t,y)\mathcal{Y}(t,x)-I(\mathcal{Y}(t,x))\mathcal{Y}(t,x)-\frac{1}{2}\|\theta\|^2\frac{\mathcal{Y}^2(t,x)}{\mathcal{Y}_x(t,x)}+U(I(\mathcal{Y}(t,x)))\right]=0$$

Substituting (6.14) in the above equation, we obtain

$$\mathcal{Y}(t,x)\left[-\mathcal{X}_t(t,y)+r\mathcal{X}(t,y)-(\beta-r+\|\theta\|^2)\mathcal{Y}(t,x)\mathcal{X}_y(t,y)-\frac{1}{2}\|\theta\|^2\mathcal{Y}^2(t,x)\mathcal{X}_{yy}(t,y)-I(\mathcal{Y}(t,x))\right]$$

But the quantity inside the square brackets is equal to (6.16), and hence equal to 0, which proves that the quantities $\widehat{\pi}$ and $\widehat{c}$ given above are optimal.

7. Portfolio Optimization with Transaction costs .

7.1. Preliminary Remarks. Constantinides and Magill [31] were the first to introduce transaction costs in the context of the Merton problem. They assume a class of utility functions and solve the problem in the discrete time case with techniques related to stochastic optimal control. They define a solvency region where the wealth of the investor should stay within. In analogy with the Merton proportion where the investor needs to trade continuously to retain his optimal stock to wealth ratio, the investor now trades in order to retain his wealth inside a region. In a follow up paper by Magill [30], the author uses results of the first paper to argue that the investment in mutual funds can be preferential to individual investment in equities because of the lower transaction costs of the former. Taksar, Klass Assaf [42] solve the continuous singular stochastic control problem of the maximization of the rate of growth of the investor's wealth, and show that it is optimal to trade within a region where its boundary points only depend on the parameter of the model and not the wealth itself. Davis and Norman [16] apply stochastic optimal control to solve the same problem and provide a detailed analysis and an algorithm to obtain the optimal policy. Shreve and Soner [41] provide an analysis based on viscosity solutions to Hamilton-Jacobi-Bellman equations developed by [10].

In the following text, we introduce the Merton problem we described in section (6), generalized by the inclusion of proportional transactions costs. We formulate the HJB equation associated with the maximization problem, and we show that the key point of the analysis is the homotheticity of the value function. The homothetic property of the power and logarithmic function allows to reduce the dimension of the value function to the one that depends only to one variable. We adopt the notation of [41], but with slight alterations to conform with the notation of the Merton problem defined above.

7.2. The Setup. In the case where financial transactions are subject to frictions, we generalize the 2-asset Merton problem described in (6.1) to include proportional transaction costs. Let $\lambda, 0 \leq \lambda < 1$ be the proportional transaction costs that applies to all transfers from the bank account for the purpose of acquiring equities, and $m, 0 \leq m < 1$ be the converse. Then, in analogy to the Merton problem formulation, we have

$$dZ(t) = rZ(t)dt - c(t)dt + (1-\mu)dM(t) - dL(t), \quad Z(0) = z \tag{7.1}$$

$$dY(t) = Y(t)(bdt + \sigma dW) + (1-\lambda)dL(t) - dM(t), \quad Y(0) = y \tag{7.2}$$

The solvency region (see figure 7.1) is defined as

$$\mathcal{Q}_{\lambda,\mu} = \{(x, y \in \mathbb{R}^2) : z + \frac{y}{1-\lambda} > 0, z + (1-\mu)y > 0\}$$

where the boundaries are given by

$$\vartheta_\lambda^- = \{y \leq 0 : z + \frac{y}{1-\lambda}\}$$

$$\vartheta_\mu^+ = \{y > 0 : z + (1-\mu)y\}$$

The investor's wealth is a reflecting diffusion inside the solvency zone, and the investor needs to consider appropriate correction actions in order to stay in the solvency region. If the investor's wealth is on the lower boundary ϑ_λ^-, then the investor needs to use cash reserves to cover the short positions in equities. On the other side, if the wealth is on ϑ_μ^+, the investor needs to sell his stock positions to cover his debt.

An investment policy is admissible if Z, Y, given by 7.1- 7.2, are in the solvency region $\mathcal{Q}_{\lambda,\mu}$. Such class of admissible strategies (c, L, M) will be denoted by $\mathcal{A}(z, y)$.

Remark. If the initial holding in cash and stock lie in the boundaries $\vartheta\mathcal{Q}_{\lambda,\mu}$, then the only admissible strategy is to return to the origin (see remark 2.1 in [41] for proof of this assertion).

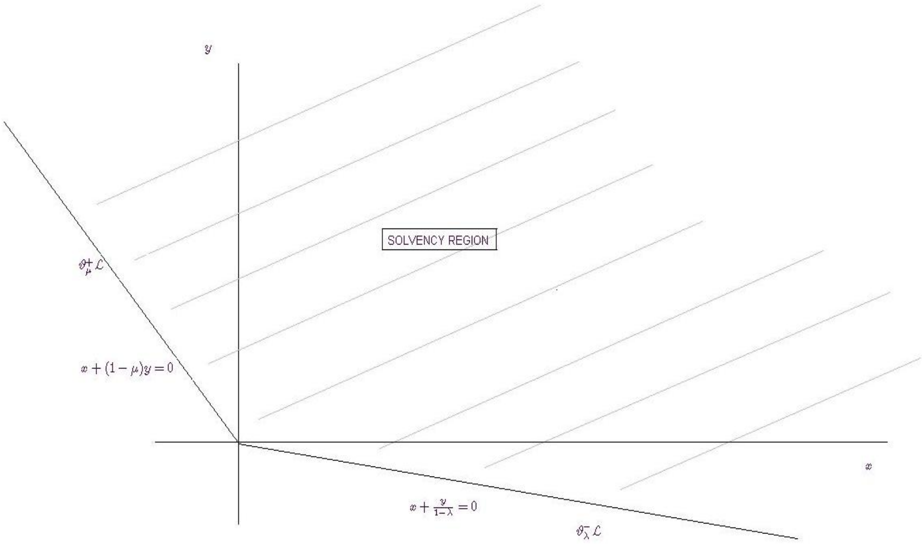

FIG. 7.1. *Solvency Region for Merton Problem with transaction costs.*

7.3. The Optimization Problem. Assume that the following condition holds throughout,

$$\sup_{c,L,M\in\mathcal{A}(z,y)} E\int_0^\infty e^{-\beta t}\max\{U(c(t)),0\}dt<\infty$$

then the value function is defined as follows

$$V(z,y)=\sup_{c,L,M\in\mathcal{A}(z,y)} E\int_0^\infty e^{-\beta t}U(c(t))dt$$

for a discount factor $\beta>0$. The utility function is defined explicitly by the following

$$U(c)=\begin{cases}\frac{c^p}{p} & \text{if } p<1,\ p\neq 0\\ \log c & \text{if } p=0\end{cases}$$

for $p<1$ and $c\geq 0$.

The value function $V(z,y)$ is concave and continuous on $\mathcal{Q}_{\lambda,\mu}$ and has the homothetic property

$$V(kz,ky)=\begin{cases}k^pV(z,y) & \text{if } p<1,\ p\neq 0\\ \frac{1}{\beta}\log k+V(z,y) & \text{if } p=0\end{cases}$$

7.4. The Dynamic Programming Equation. Let $G(z,y)$ in $C^2(\mathcal{Q}_{\lambda,\mu})$ and define the second order differential operator by

$$\mathcal{L}G(z,y)=\beta G(z,y)-rxG_z(z,y)-byG_y(z,y)-\frac{1}{2}\sigma^2y^2G_{yy}(z,y)$$

Then, the Hamilton-Jacobi-Bellman equation is given by

$$\min\{\mathcal{L}G(z,y)-\widetilde{U}(G_z(z,y)),-(1-\mu)G_z(z,y)+G_y(z,y),G_z(z,y)-(1-\lambda)G_y(x,y)\}$$

where $\widetilde{U}$ is the convex dual of U presented in (4.1) above.

Observe that from the homotheticity property of the value function, we have

$$V(kz,ky)=k^pV(z,y)$$

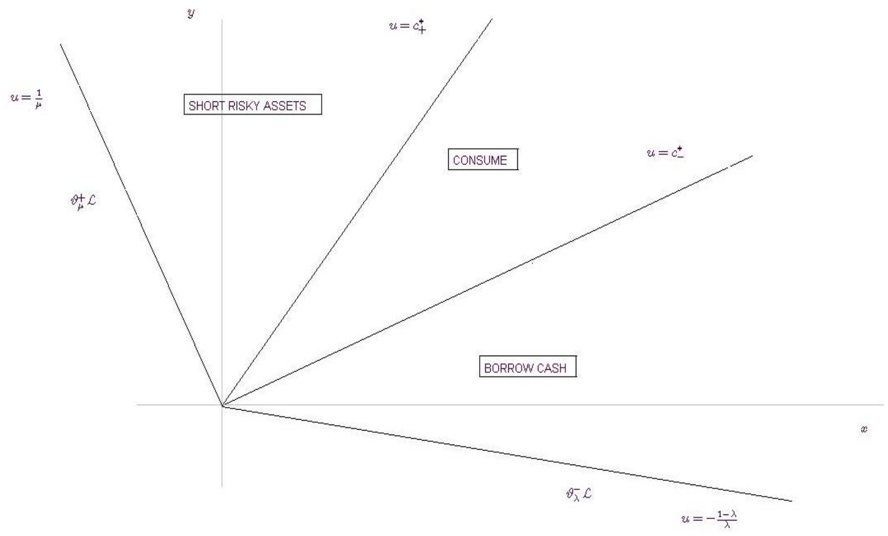

FIG. 7.2. *Optimal Consumption Region for Merton Problem with transaction costs.*

and by applying the transformation

$$f(u) := V(1-u,u), \quad -\frac{1-\lambda}{\lambda} \le u \le \frac{1}{\mu}$$

we can represent the value function with one only variable

$$V(z,y) = (z+y)^p f(u), \quad u = \frac{y}{z+y}$$

in order to solve the new dynamic programming equation by standard numerical methods.

It follows that there are points

$$-\frac{1-\lambda}{\lambda} \le c_+^* \le \pi^* \le c_-^* \le \frac{1}{\mu}$$

that optimal consumption $c^* = (V_z(z,y))^{\frac{1}{p-1}}$ is achieved inside a wedge (see figure 7.2) when

$$\mathcal{L}V(z,y) - \widetilde{U}(V_z(z,y)) = 0, \qquad c_-^* \le u \le c_+^*$$

Moreover, the "short risky assets" region satisfies

$$-(1-\mu)V_z(z,y) + V_y(z,y) = 0, \qquad c_+^* \le u \le \frac{1}{\mu}$$

and the "borrow cash" region satisfies

$$V_z(z,y) - (1-\lambda)V_y(x,y) = 0, \qquad -\frac{1-\lambda}{\lambda} \le u \le c_-^*$$

where the constants c_+^* and c_-^* can be calculated numerically.

8. The General Semimartingale Problem .

8.1. Preliminary Remarks. In this setting, we consider an agent who wants to maximize the expected utility of his terminal wealth. We operate in a finite probability space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$, $1 \le n \le N$ and we find a solution to the dual variational problem and subsequently determine the solution to the primal using convex duality theory. Our review is based mainly on the works of Kramkov, Schachermayer [26] and Schachermayer [40].

8.2. The Financial Market. Assume an agent who has an initial endowment $x > 0$ and wishes to invest in $d+1$ assets, and whose investment plans are confined within a finite investment horizon T. The assets are d stocks $S_1, S_2, \dots, S_d$ and one bond S_0. The bond is taken as numeraire and without generality loss, we assume that $S_0 = 1$. We define the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$ and assume that the process S is adapted to $\mathcal{F}_t$, where S now represent the vector of the $d+1$ price processes.

Let $\pi_i,\ 1 \leq i \leq d$ be a predictable S-integrable investment strategy correponding the the i-th asset, where by predictability we mean that the π_t strategy is $\mathcal{F}_{t-1}$ adapted. Then, we can define a self-financing portfolio Π, characterized by the pair (x, π), and whose value at time t is given by:

$$X_t = x + (\pi \cdot S)_t = x + \int_0^T \pi_s dS_s, \qquad 0 \leq t \leq T$$

The process $(\pi \cdot S)_t$ is called admissible if is bounded below by a constant $k \in \mathbb{R}_+$ such that

$$(\pi \cdot S)_t \geq -k$$

holds. The above requirement is necessary to forbid investor from over-leveraging to cover shortfalls from strategies such as doubling (see [20]).

In the discrete time case, the stochastic integral reduces to

$$(\pi \cdot S)_t = \sum_{s=1}^{t} \pi_s (S_s - S_{s-1})$$

and the admissibility criterion, defined above, translates to

$$\sum_{s=1}^{t} \pi_s \Delta S_s \geq -k$$

The set of equivalent martingale measures $\widetilde{\mathbf{P}}$ of $\mathbf{P}$ that turn the stochastic integral $(\pi \cdot S)_t$ into a local martingale under $\widetilde{\mathbf{P}}$ is denoted by $\mathcal{M}$. For the complete market case, the set $\mathcal{M}$ reduces to a singleton.

8.3. Solution to the Optimization Problem in a Finite Probability Space. We are interested in maximizing the expected utility of terminal wealth of the agent subject to the budget constraint

$$\text{Maximize} \qquad E_{\mathbf{P}}[X_T] = \sum_{n=1}^{N} p_n U(\xi_n)$$

$$\text{s.t} \qquad E_{\widetilde{\mathbf{P}}}[X_T] = \sum_{n=1}^{N} \widetilde{p}_n \xi_n \leq x$$

where $\xi_n = X_T(\omega_n)$. The utility is strictly concave, defined in section (4), and could intuitively interpreted as the diminishing appetite of the investor for additional wealth. The constrain could be economically interpreted as a requirement from an investor to cover liabilities, for example to cover a contingent claim or to receive a pension lump sum amount at time T. It could could also be interpreted as the requirement of a fund to cover expected redemptions at time T. Moreover, the monotonicity, concavity, and continuity of the utility facilitates the formulation of existence and uniqueness results through Krein, Milman theorem which ensures at least one extreme point (see [29, 38]). **Assumption**. The utility function U satisfies the asymptotic elasticity property

$$\lim_{x \to \infty} \sup \frac{xU'(x)}{U(x)}$$

The above is necessary and sufficient condition for a unique optimal solution to the optimization problem. The ratio of the marginal to the average utility should be smaller than one, which means that asymptotically the marginal utility becomes smaller that the average utility, which confirms the concavity of the utility function.

We turn the constrained problem to unconstrained, with the help of Lagrange multipliers

$$L(\xi, y) = \sum_{n=1}^{N} p_n U(\xi_n) - y\left(\sum_{n=1}^{N} \widetilde{p}_n \xi_n - x\right)$$

$$yx + \sum_{n=1}^{N} p_n \left(U(\xi_n) - \xi_n y \frac{\widetilde{p}_n}{p_n}\right)$$

The objective is to find $\widehat{\xi}$ that satisfies

$$\Psi(y) = \sup_{\xi} L(\xi, y), \qquad y \geq 0$$

for a unique $\widehat{y}$.

By using the conjugate $\widetilde{U}$ of the utility U (see section (4)), the dual problem can be written as

$$\Psi(y) = \sum_{n=1}^{N} p_n \widetilde{U}\left(y \frac{\widetilde{p}_n}{p_n}\right) + yx$$

and conclude that $\Psi'(\widehat{y}) = -x$ is the unique minimizer of the dual problem.

We deduce that the maximum point in the Lagrangian is achieved when

$$\widehat{\xi}_n = I\left(\widehat{y}\frac{\widetilde{p}_n}{p_n}\right)$$

In particular, $\widehat{\xi}_n$ saturates the budget constraint inequality, and we obtain the maximal primal value function

$$V(x) = \sum_{n=1}^{N} p_n U(\widehat{\xi}_n)$$

and the minimal dual by

$$\widetilde{V}(y) = \sum_{n=1}^{N} p_n \widetilde{U}(\widehat{y}\frac{\widetilde{p}_n}{p_n})$$

Appendix A. Feynman-Kac.

A.1. Feynman-Kac and HJB equations. For completeness, we state the Feynman-Kac theorem without proof. The reader may refer to [**?** 24] for related proofs.

THEOREM A.1. *(Feynman-Kac)*
Let $V \in C^2(\mathbb{R}^k)$ and define the infinitesimal generator by,

$$\mathcal{L}_t V(x) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}\alpha^{(ij)}(t,x)\frac{\vartheta^2 V(x)}{\vartheta x_i x_j} + \sum_{i=1}^{k} b_i(t,x)\frac{\vartheta V(x)}{\vartheta x_i} \tag{A.1}$$

Assume $V(x)$ satisfies the Cauchy problem

$$\frac{\vartheta V}{\vartheta t} + \mathcal{L}_t V - hV + g = 0$$

where $f : R \to [0,\infty)$ and $g, h : \mathcal{S} \times \mathbb{R}^k$
and satisfy the following growth conditions

$$|f(x)| \leq L(1 + \|X\|^{2\lambda})$$

$$|g(t,x)| \leq L(1 + \|X\|^{2\lambda})$$

for constants $L > 0$, $\lambda \geq 1$
with terminal condition

$$V(T,x) = f(x), x \in \mathbb{R}$$

Then, $V(t,x)$ admits a stochastic representation

$$V(t,x) = E^{t,x}\left[f(X(T))\exp\{-\int_t^T h(s,X_s)ds\} + \int_t^T g(u,X_u)\exp\{-\int_t^T h(s,X_s)ds\}du\right] \tag{A.2}$$

Let

$$X_s^{t,x} = x + \int_t^T b(s,X_s,\pi)ds + \int_t^T \sigma(s,X_s,\pi)dW_s$$

be a stochastic differential equation, where π is the process we want to control, and set

$$V(t,x) = \sup E^{t,x}\left[\int_t^T g(s,X_s,\pi)ds + f(X(T))\right]$$

as the value function we want to maximize over some suitable functions f,g.
Then, $V(t,x)$ solves the Hamilton-Jacobi-Bellman equation

$$\frac{\vartheta V}{\vartheta t} + \sup[\mathcal{L}_t V + g] = 0$$

for the terminal condition

$$V(T,x) = f(x), \quad x \in \mathbb{R}$$

Appendix B.

[1] J. P. Aubin. Mathematical methods of game and economic theory. *North-Holland Publ.Co., N.Y., 1979, 620*, 1979.

[2] A. Bensoussan. *Stochastic control of partially observable systems.* Cambridge University Press New York, NY, USA, 1992.

[3] D. P. Bertsekas. *Dynamic Programming and Stochastic Control.* Academic Press, Inc. Orlando, FL, USA, 1976.

[4] D. P. Bertsekas. *Dynamic programming and optimal control.* Athena Scientific Belmont, Mass, 1995.

[5] D. P. Bertsekas, A. Nedic, and A. E. Ozdaglar. *Convex analysis and optimization.* Athena Scientific Belmont, Mass, 2003.

[6] J. M. Bismut. Conjugate convex functions in optimal stochastic control. *J.Math.Anal.Appl*, 44(1973):384–404, 1973.

[7] J. M. Bismut. Growth and optimal intertemporal allocation of risks. *Journal of Economic Theory*, 10:239287, 1975.

[8] J. M. Bismut. An introductory approach to duality in optimal stochastic control. *SIAM Review*, 20(1):62–78, 1978.

[9] J. C. Cox and C. Huang. *A Variational Problem Arising in Financial Economics.* Sloan School of Management, Massachusetts Institute of Technology, 1988.

[10] M. G. Crandall and P. L. Lions. Viscosity solutions of hamilton-jacobi equations. *Transactions of the American Mathematical Society*, 277(1):1–42, 1983.

[11] D. Cuoco. Optimal consumption and equilibrium prices with portfolio constraints and stochastic income. *Journal of Economic Theory*, 72(1):33–73, 1997.

[12] J. Cvitanic and I. Karatzas. Convex duality in constrained portfolio optimization. *The Annals of Applied Probability*, 2(4):767–818, 1992.

[13] J. Cvitanic and I. Karatzas. Hedging and portfolio optimization under transaction costs: a martingale approach. *Mathematical Finance*, 6(2):133165, 1996.

[14] J. Cvitanic, H. Pham, and N. Touzi. A closed-form solution to the problem of super-replication under transaction costs. *Finance and Stochastics*, 3(1):35–54, 1999.

[15] M. H. A. Davis. A problem of optimal investment with randomly terminating income. Technical report, 2003.

[16] M. H. A. Davis and AR Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676–713, 1990.

[17] I. Ekeland and R. Temam. Convex analysis and variational problems [russian translation]. *Mir, Moscow*, 1979.

[18] W. H. Fleming and R. W. Rishel. *Deterministic and stochastic optimal control.* Springer, 1975.

[19] W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions.* Springer-Verlag Gmbh, 1992.

[20] J. M. Harrison and S. R. Pliska. Martingales and stochastic integrals in the theory of continous trading. *Stochastic Processes and their Applications*, 11:215–260, 1981.

[21] Y. M. Kabanov and G. Last. Hedging under transaction costs in currency markets: a continuous-time model. *Mathematical Finance*, 12(1):63–70, 2002.

[22] I. Karatzas, J. P. Lehoczky, and S. E. Shreve. Optimal portfolio and consumption decisions for a small investor on a finite horizon. *SIAM Journal on Control and Optimization*, 25(6):1557–1586, 1987.

[23] I. Karatzas, J. P. Lehoczky, S. E. Shreve, and G. L. Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization*, 29(3):702–730, 1991.

[24] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus.* Springer-Verlag New York, 1991.

[25] I. Karatzas and S. E. Shreve. *Methods of Mathematical Finance.* Springer, 1998.

[26] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability*, 9(3):904–950,

1999.

[27] N. V. Krylov. *Controlled diffusion processes.* Springer-Verlag New York, 1980.

[28] P. Lakner. Optimal trading strategy for an investor: the case of partial information. *Stochastic Processes and their Applications*, 76(1):77–97, 1998.

[29] P. D. Lax. *Functional analysis.* Wiley New York, 2002.

[30] M. J. P. Magill. The preferability of investment through a mutual fund. *Journal of Economic Theory*, 13:264–271, 1976.

[31] M. J. P. Magill and G. M. Constantinides. Portfolio selection with transaction costs. *Journal of Economic Theory*, 13(2):245263, 1976.

[32] H. Markowitz. Portfolio selection. *The Journal of Finance*, 7(1):77–91, 1952.

[33] R. C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of economics and statistics*, 51(3):247–257, 1969.

[34] R. C. Merton. *Optimum Consumption and Portfolio Rules in a Continuous-time Model.* MIT, 1970.

[35] H. Pham and M. C. Quenez. Optimal portfolio in partially observed stochastic volatility models. *The Annals of Applied Probability*, 11(1):210–238, 2001.

[36] R. T. Rockafellar. Conjugate convex functions in optimal control and the calculus of variations. *J.Math.Anal.Appl.*, 32:174–222, 1970.

[37] R. T. Rockafellar. *Convex Analysis.* Princeton University Press, 1996.

[38] W. Rudin. *Functional analysis.* McGraw-Hill New York, 1973.

[39] P. A. Samuelson. Lifetime portfolio selection by dynamic stochastic programming. *The review of economics and statistics*, 51(3):239–246, 1969.

[40] W. Schachermayer. Optimal investment in incomplete financial markets. *Mathematical Finance: Bachelier Congress*, 2000:427462, 2000.

[41] S.E. Shreve and H.M. Soner. Optimal investment and consumption with transaction costs. *The Annals of Applied Probability*, 4(3):609–692, 1994.

[42] M. Taksar, M. J. Klass, and D. Assaf. A diffusion model for optimal portfolio selection in the presence of brokerage fees. *Mathematics of Operations Research*, 13(2):277–294, 1988.

[43] G. L. Xu and S. E. Shreve. A duality method for optimal consumption and investment under short-selling prohibition. i. general market coefficients. *The Annals of Applied Probability*, 2(1):87–112, 1992.

[44] Thaleia Zariphopoulou. Investment-consumption models with transaction fees and markov-chain parameters, 1992.